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Simplicial blowups and discrete normal surfaces in `simpcomp`

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Abstract

`simpcomp` is an extension to `GAP`, the well known system for computational discrete algebra. It allows the user to work with simplicial complexes. In the latest version, support for *simplicial blowups* and *discrete normal surfaces* was added. Furthermore, new functions for constructing certain infinite series of triangulations have been implemented and interfaces to other software packages have been improved to previous versions.

1 Introduction

`simpcomp` [8, 7] is a package for working with simplicial complexes. Its aim is to provide the user with a broad spectrum of functionality regarding simplicial constructions and the calculation of properties of simplicial complexes.

Important goals during the development of `simpcomp` were interactivity, ease of use, completeness of documentation and ease of extensibility. The software allows the user to interactively construct simplicial complexes and to compute their properties in the `GAP` [9] or `SAGE` [14] shell. It is sought of as a tool for researchers to verify or disprove a conjecture one might have and to quickly do simplicial constructions using the computer. Furthermore, it makes use of `GAP`'s expertise in groups and group operations. For example, automorphism groups (cf. [2]) and fundamental groups of simplicial complexes can be computed and examined further within the `GAP` system.

With its development being started in March 2009 as part of the PhD theses [6] and [18], `simpcomp` still is a rather young project, but now already contains roughly 270 functions and its manual [7] contains about 180 pages of documentation. At the ISSAC 2010 in Munich, `simpcomp` won the *Best Software Presentation Award* by the Fachgruppe Computeralgebra.

Since then, a lot of mathematical functionality was added, namely the support for simplicial blowups and discrete normal surfaces. In the following, we will give a short overview about these features. For a brief introduction into `simpcomp`'s basic functionality and design principles see [8].

2 New features in Version 1.5

Simplicial blowups: In algebraic geometry, blowups provide a useful way to study singularities of algebraic varieties [10]. The idea is to replace a point by all lines passing through that point. This concept is now available for combinatorial 4-manifolds (cf. [16]) and integrated into `simpcomp`.

Discrete normal surfaces [17] and *slicings*: Slicings of combinatorial d-manifolds are (non-singular) $(d-1)$-dimensional level sets of polyhedral Morse functions. In dimension 3, slicings are discrete normal

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surfaces. simpcomp supports discrete normal surfaces as a new object type and enables the user to generate and analyze slicings together with the corresponding Morse functions.

In addition, new infinite series of highly symmetric triangulations have been integrated. Some of them were just recently found by the second author. In particular, simpcomp contains the first computer implementation of these series presented in [18, Chapter 4].

Finally, simpcomp now can use the GAP package homalg [4] for its homology computations. This allows the computation of (co-)homology groups of simplicial complexes over arbitrary rings and fields, as well as the usage of all the functionality related to homological algebra that homalg provides.

3 Roadmap

The current version of simpcomp is 1.5. On the roadmap for the upcoming versions are a closer interaction with other software packages in the field (e.g. Macaulay2 [11]), faster bistellar moves and the implementation of a combinatorial formula for calculating the Stiefel-Whitney class of combinatorial manifolds [3] due to Banchoff.

Furthermore, as a long term goal, we would like to provide the functionality to perform surgery on combinatorial 3- and 4-manifolds. This would be a step forward to constructing candidates for combinatorial manifolds with exotic PL structures as already done in the smooth setting by Akbulut [1].

4 Examples

This section contains a small demonstration of the capabilities of simpcomp in form of transcripts of the GAP shell for some example constructions. Most of the features presented below have been newly introduced in the versions 1.4 and 1.5.

4.1 Normal surfaces in cyclic 4-polytopes.

For $n \geq 3$, consider the cyclic 4-polytope $C_4(2n)$ on $2n$ vertices with vertex labels 1 to n. By Gale's evenness condition, neither the span of all odd nor the span of all even vertices in $C_4(2n)$ contains a triangle of $C_4(2n)$. Thus, given the combinatorial 3-sphere $S = \partial C_4(2n)$, a level set of a Morse function on S separating the even from the odd vertices gives rise to a handle body decomposition of S — this is a discrete normal surface in the sense of [17].

This construction can be done in simpcomp as follows. Note that we arbitrarily chose $n = 5$ for demonstration purposes below.

```
gap> LoadPackage("simpcomp");; #load the package
Loading simpcomp 1.4.0
by F.Effenberger and J.Spreer
http://www.igt.uni-stuttgart.de/LstDiffgeo/simpcomp
gap> c_4_10:=SCBdCyclicPolytope(4,10);;
```

Above, we constructed the boundary of the cyclic 4-polytope $\partial C_4(10)$ on 10 vertices. We now look at the level set of a Morse function on c_4_10 which separates even and odd vertices:

```
gap> sl:=SCSlicing(c,[[1,3,5,7,9],[2,4,6,8,10]]);
[NormalSurface

 Properties known: Chi, ConnectedComponents, ... , VertexLabels, Vertices.

 Name="slicing [ [ 1, 3, 5, 7, 9 ], [ 2, 4, 6, 8, 10 ] ] of Bd(C_4(10))"
 Dim=2
 Chi=-10
 F=[ 25, 70, 0, 35 ]
 IsConnected=true
 TopologicalType="(T^2)#6"

/NormalSurface]
```

The resulting polytopal complex on 25 vertices is a discrete normal surface without triangles and with 35 quadrilaterals. Topologically, it is the orientable surface with Euler characteristic -10, and thus homeomorphic to $(\mathbb{T}^2)^{\#6}$. As a next step we now could triangulate the normal surface and do further examinations using `simpcomp`'s standard functionality.

4.2 Combinatorial blowups of the Kummer variety K^4

The 4-dimensional abstract Kummer variety K^4 with 16 nodes leads to the $K3$ surface by resolving the 16 singularities [15]. Using `simpcomp`, this process can be carried out in a combinatorial setting, cf. [16]. The first step of this so-called *dilatation* or *blowup process* can be done as follows.

We first load the singular 16-vertex triangulation of K^4 due to Kühnel [12] from the library.

```
gap> SCLib.SearchByName("Kummer");
[ [ 7493, "4-dimensional Kummer variety (VT)" ] ]
gap> k4:=SCLib.Load(7493);;
```

We now verify that the link of vertex 1 in K^4 topologically is a real projective 3-space. The ranks of its integral homology groups and its fundamental group are the following:

```
gap> lk1:=k4.Link(1);;
gap> lk1.Homology;
[ [ 0, [  ] ], [ 0, [ 2 ] ], [ 0, [  ] ], [ 1, [  ] ] ]
gap> pi:=lk1.FundamentalGroup;
<fp group with 61 generators>
gap> Size(pi);
2
```

We now verify that, as suspected, the complex is PL homeomorphic to the minimal 11-vertex triangulation of $\mathbb{R}P^3$ from the library. This is done using a heuristic algorithm based on bistellar moves [5].

```
gap> SCLib.SearchByName("RP^3");
[ [ 45, "RP^3" ], [ 113, "RP^3=L(2,1) (VT)" ], ... ]
gap> minRP3:=SCLib.Load(45);;
gap> SCEquivalent(lk1,minRP3);
#I  SCReduceComplexEx: complexes are bistellarly equivalent.
true
```

Finally, we resolve the singularity of K^4 at vertex 1 by a simplicial blowup.

```
gap> c:=SCBlowup(k4,1);;
...
#I  SCBlowup: ...blowup completed.
#I  SCBlowup: You may now want to reduce the complex via 'SCReduceComplex'.
```

Indeed, the second Betti number increased by 1, again as expected.

```
gap> k4.Homology;
[ [ 0, [  ] ], [ 0, [  ] ], [ 6, [ 2, 2, 2, 2, 2 ] ], [ 0, [  ] ], [ 1, [  ] ] ]
gap> c.Homology;
[ [ 0, [  ] ], [ 0, [  ] ], [ 7, [ 2, 2, 2, 2 ] ], [ 0, [  ] ], [ 1, [  ] ] ]
```

The resulting complex now only has 15 singularities. By iterating this process 15 more times, we obtain a combinatorial triangulation of the $K3$ surface with standard PL structure.

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